

## Chapter 17

# Algebraic Geometry for CAGD

Initially, the field of computer aided geometric design and graphics drew most heavily from differential geometry, approximation theory, and vector geometry. Since the early 1980's, some of the tools of algebraic geometry have been introduced into the CAGD literature. Much, though not all, of modern algebraic geometry deals with abstractions which are far removed from the algorithmic nature of computer aided geometric design and graphics, whereas mathematicians of 50-150 years ago tended to write in less abstract terms. This chapter gleans some concepts from 19<sup>th</sup> century classical algebraic geometry that answer some basic questions that are pertinent to CAGD.

1. Given a planar curve defined parametrically as  $x = \frac{x(t)}{w(t)}$ ,  $y = \frac{y(t)}{w(t)}$  where  $x(t)$ ,  $y(t)$ , and  $w(t)$  are polynomials, find an implicit equation  $f(x, y) = 0$  which defines the same curve. This process of parametric to implicit conversion will be referred to as *implicitization*.
2. Given the  $(x, y)$  coordinates of a point which lies on a parametric curve  $x = \frac{x(t)}{w(t)}$ ,  $y = \frac{y(t)}{w(t)}$ , find the parameter value  $t$  which corresponds to that point. This problem will be referred to as the *inversion* problem.
3. Compute the points of intersection of two parametric curves using the implicitization and inversion techniques.

Section 1.4 presents some preliminary terminology and theorems. Sections 17.1 through 17.4 discuss the implicitization and inversion of planar curves, and Section 17.5 applies those tools to computing curve intersections. Section 17.7 discusses some special properties of parametric cubic curves and Section 17.8 overviews surface implicitization. Section 17.11 discusses Gröbner bases.

### 17.1 Implicitization

It was noted that there are basically two ways that a planar curve can be defined: parametrically ( $x = x(t)/w(t)$ ,  $y = y(t)/w(t)$ ) and implicitly ( $f(x, y) = 0$ ).

Obviously, the parametric equation of a curve has the advantage of being able to quickly compute the  $(x, y)$  coordinates of several points on the curve for plotting purposes. Also, it is simple to define a curve *segment* by restricting the parameter  $t$  to a finite range, for example  $0 \leq t \leq 1$ . On the other hand, the implicit equation of a curve enables one to easily determine whether a given point lies on the curve, or if not, which side of the curve it lies on.

Given these two different equations for curves, it is natural to wonder if it is possible to convert between representations for a given curve. The answer is that it is *always* possible to find an implicit equation of a parametric curve, but a parametric equation can generally be found only for implicit curves of degree two or one. The process of finding the implicit equation of a curve which is expressed parametrically is referred to as *implicitization*. In Section 17.4, we will discuss how this can be accomplished using an important algebraic tool, the *resultant*, and Section 17.3 discusses resultants. For fun, Section 17.2 illustrates how someone might tackle the implicitization problem before learning about resultants. Section 17.5 applies these ideas to the problem of intersecting two parametric curves.

## 17.2 Brute Force Implicitization

Consider this simple example of parametric-to-implicit conversion: Given a line

$$x = t + 2 \quad y = 3t + 1,$$

we can easily find an implicit equation which identically represents this line by solving for  $t$  as a function of  $x$

$$t = x - 2$$

and substituting into the equation for  $y$ :

$$y = 3(x - 2) + 1$$

or  $3x - y - 5 = 0$ . Note that this implicit equation defines *precisely* the same curve as does the parametric equation. We can also identify two inversion equations (for finding the parameter value of a point on the line):  $t = x - 2$  or  $t = (y - 1)/3$ .

This approach to implicitization also works for degree two parametric curves. Consider the parabola

$$x = t^2 + 1 \quad y = t^2 + 2t - 2.$$

Again, we can solve for  $t$  as a function of  $x$ :

$$t = \pm\sqrt{x - 1}$$

and substitute into the equation for  $y$ :

$$y = (\sqrt{x - 1})^2 \pm 2\sqrt{x - 1} - 2.$$

We can isolate the radical and square both sides

$$(y - (x - 1) + 2)^2 = (\pm 2\sqrt{x - 1})^2$$

to yield

$$x^2 - 2xy + y^2 - 10x + 6y + 13 = 0$$

which is the desired implicit equation. Again, this implicit equation defines exactly the same curve as does the parametric equation.

We run into trouble if we try to apply this implicitization technique to curves of degree higher than two. Note that the critical step is that we must be able to express  $t$  as a function of  $x$ . For cubic and quartic equations, this can be done, but the resulting expression is hopelessly complex. For curves of degree greater than four, it is simply not possible.

We cannot obtain an inversion equation for this parabola the way we did for the straight line. For example, suppose we want to find the parameter of the point  $(5, -2)$  which we know to lie on the curve. The brute force approach would be to find the values of  $t$  which satisfy the equation

$$x = 5 = t^2 + 1$$

and then to compare them with the values of  $t$  which satisfy the equation

$$y = -2 = t^2 + 2t - 2.$$

In the first case, we find  $t = -2$  or  $2$ , and in the second case,  $t = -2$  or  $0$ . The value of  $t$  which satisfies both equations is  $-2$ , which must therefore be the parameter value of the point  $(5, -2)$ .

This unsuccessful attempt at implicitization and inversion motivates the following discussion of *resultants*, which will provide an elegant, general solution to the implicitization and inversion problems.

### 17.3 Resultants

Resultants address the question of whether two polynomials have a common root. Consider the two polynomials

$$f(t) = \sum_{i=0}^n a_i t^i \quad g(t) = \sum_{i=0}^n b_i t^i$$

The resultant of  $f(t)$  and  $g(t)$ , written  $R(f, g)$ , is an expression in terms of the coefficients  $a_i$  and  $b_i$  such that a common root of  $f(t)$  and  $g(t)$  exists if and only if  $R(f, g) = 0$ .

We illustrate by finding the resultant of two cubic polynomials

$$f(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0 \quad g(t) = b_3 t^3 + b_2 t^2 + b_1 t + b_0.$$

In other words, we want to determine whether there exists a value  $\alpha$  such that  $f(\alpha) = g(\alpha) = 0$  without having to actually find all roots of both polynomials and comparing. We begin by forming three auxiliary polynomials  $h_1(t)$ ,  $h_2(t)$  and  $h_3(t)$  as follows:

$$\begin{aligned} h_1(t) &= a_3 g(t) - b_3 f(t) \\ &= (a_3 b_2) t^2 + (a_3 b_1) t + (a_3 b_0) \end{aligned}$$

where  $(a_i b_j) \equiv (a_i b_j - a_j b_i)$  and

$$\begin{aligned} h_2(t) &= (a_3 t + a_2) g(t) - (b_3 t + b_2) f(t) \\ &= (a_3 b_1) t^2 + [(a_3 b_0) + (a_2 b_1)] t + (a_2 b_0) \end{aligned}$$

$$\begin{aligned} h_3(t) &= (a_3 t^2 + a_2 t + a_1) g(t) - (b_3 t^2 + b_2 t + b_1) f(t) \\ &= (a_3 b_0) t^2 + (a_2 b_0) t + (a_1 b_0) \end{aligned}$$

Note that if there exists a value  $\alpha$  such that  $f(\alpha) = g(\alpha) = 0$ , then  $h_1(\alpha) = h_2(\alpha) = h_3(\alpha) = 0$ . We can therefore say that  $f(t)$  and  $g(t)$  have a common root if and only if the set of equations

$$\begin{bmatrix} (a_3 b_2) & (a_3 b_1) & (a_3 b_0) \\ (a_3 b_1) & (a_3 b_0) + (a_2 b_1) & (a_2 b_0) \\ (a_3 b_0) & (a_2 b_0) & (a_1 b_0) \end{bmatrix} \begin{Bmatrix} t^2 \\ t \\ 1 \end{Bmatrix} = 0$$

has a solution.<sup>1</sup> However, we know from linear algebra that this set of homogeneous linear equations can have a solution if and only if

$$\begin{vmatrix} (a_3b_2) & (a_3b_1) & (a_3b_0) \\ (a_3b_1) & (a_3b_0) + (a_2b_1) & (a_2b_0) \\ (a_3b_0) & (a_2b_0) & (a_1b_0) \end{vmatrix} = 0$$

and therefore,

$$R(f, g) = \begin{vmatrix} (a_3b_2) & (a_3b_1) & (a_3b_0) \\ (a_3b_1) & (a_3b_0) + (a_2b_1) & (a_2b_0) \\ (a_3b_0) & (a_2b_0) & (a_1b_0) \end{vmatrix}$$

This same approach can be used to construct the resultant of polynomials of any degree.

Let's try this resultant on a couple of examples. First, let  $f(t) = t^3 - 2t^2 + 3t + 1$  and  $g(t) = 2t^3 + 3t^2 - t + 4$ . For this case,

$$R(f, g) = \begin{vmatrix} 7 & -7 & 2 \\ -7 & -5 & -11 \\ 2 & -11 & 13 \end{vmatrix} = -1611$$

We aren't so much interested in the actual numerical value of the resultant, just whether it equals zero or not. In this case,  $R(f, g) = -1611 \neq 0$ , so we conclude that  $f(t)$  and  $g(t)$  do *not* have a common root.

Consider next the pair of polynomials  $f(t) = t^3 - t^2 - 11t - 4$  and  $g(t) = 2t^3 - 7t^2 - 5t + 4$ . In this case,

$$R(f, g) = \begin{vmatrix} -5 & 17 & 12 \\ 17 & -60 & -32 \\ 12 & -32 & -64 \end{vmatrix} = 0$$

Since  $R(f, g) = 0$ ,  $f(t)$  and  $g(t)$  *do* have a common root. Note that the resultant simply determines the existence or non-existence of a common root, but it does not directly reveal the value of a common root, if one exists. In fact, if the resultant is zero, there may actually be several common roots. The next section discusses how to compute the common root(s).

## 17.4 Determining the Common Root

We present two basic approaches to finding the common root of two polynomials: by solving a set of linear equations, or by using Euclid's algorithm.

**Linear Equation Approach** Our intuitive development of the resultant of two cubic polynomials led us to a set of three linear equations in three "unknowns":  $t^2$ ,  $t$  and 1. In general, we could create the resultant of two degree  $n$  polynomials  $f(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$ ,  $g(t) = b_n t^n + b_{n-1} t^{n-1} + \dots + b_1 t + b_0$ , as the determinant of the coefficient matrix of  $n$  homogeneous linear

<sup>1</sup>Actually, we have only shown that this is a necessary condition. The proof that it is also sufficient can be found in [GSA84].

equations:

$$\begin{bmatrix} (a_n b_{n-1}) & \cdot & \cdot & \cdot & (a_n b_0) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a_n b_0) & \cdot & \cdot & \cdot & (a_1 b_0) \end{bmatrix} \begin{Bmatrix} t^{n-1} \\ t^{n-2} \\ \cdot \\ \cdot \\ t \\ 1 \end{Bmatrix} = 0$$

It may be a bit confusing at first to view this as a set of homogeneous *linear* equations, since the unknowns are all powers of  $t$ . Let us temporarily rename our unknowns:

$$\begin{bmatrix} (a_n b_{n-1}) & \cdot & \cdot & \cdot & (a_n b_0) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a_n b_0) & \cdot & \cdot & \cdot & (a_1 b_0) \end{bmatrix} \begin{Bmatrix} X_{n-1} \\ X_{n-2} \\ \cdot \\ \cdot \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

where  $X_i = t^i$ . After solving for the  $X_i$ , the common root of  $f(t)$  and  $g(t)$  can be obtained as  $t = X_{i+1}/X_i$ .

**Cramer's Rule** There are several well known methods for solving for the  $X_i$ . One way is to apply Cramer's rule. A non-trivial solution exists (that is, a solution other than  $X_0 = X_1 = \dots = X_{n-1} = 0$ ) only if the determinant of the matrix is zero. But, that implies that the  $n$  equations are linearly dependent and we can discard one of them without losing any information. We discard the *last* equation, and can then solve for  $n - 1$  homogeneous equations in  $n$  homogeneous unknowns using Cramer's rule. It turns out that occasionally we run into trouble if we discard an equation other than the last one. We illustrate Cramer's rule for the case  $f(t) = t^3 - t^2 - 11t - 4$  and  $g(t) = 2t^3 - 7t^2 - 5t + 4$ . Recall that this is the pair for which we earlier found that  $R(f, g) = 0$ . We have the set of equations

$$\begin{bmatrix} -5 & 17 & 12 \\ 17 & -60 & -32 \\ 12 & -32 & -64 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

Discarding the last equation, we obtain

$$\begin{bmatrix} -5 & 17 & 12 \\ 17 & -60 & -32 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

from which we find the common root using Cramer's rule:

$$t = \frac{X_1}{X_0} = - \frac{\begin{vmatrix} -5 & 12 \\ 17 & -32 \end{vmatrix}}{\begin{vmatrix} -5 & 17 \\ 17 & -60 \end{vmatrix}} = 4$$

**Gauss Elimination** A numerically superior algorithm for solving this set of equations is to perform Gauss elimination. Two other advantages of Gauss elimination are that it can be used to determine whether the determinant is zero to begin with, and also it reveals *how many* common roots there are. We will illustrate this approach with three examples, using integer preserving Gauss elimination. We choose the integer preserving Gauss elimination because then the lower right hand element of the upper triangular matrix is the value of the determinant of the matrix.

### Example 1

Our first example is one we considered earlier:  $f(t) = t^3 - 2t^2 + 3t + 1$  and  $g(t) = 2t^3 + 3t^2 - t + 4$ . We set up the following set of linear equations, and triangularize the matrix using integer preserving Gauss elimination:

$$\begin{bmatrix} 7 & -7 & 2 \\ -7 & -5 & -11 \\ 2 & -11 & 13 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} =$$

$$\begin{bmatrix} 7 & -7 & 2 \\ 0 & -84 & 0 \\ 0 & 0 & -1611 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

We observe that the only solution to this set of equations is  $X_2 = X_1 = X_0 = 0$ , and conclude that  $f(t)$  and  $g(t)$  do not have a common root. Note that the lower right element  $-1611$  is the determinant of the original matrix, or the resultant.

### Example 2

We next examine the pair of polynomials  $f(t) = t^3 - t^2 - 11t - 4$  and  $g(t) = 2t^3 - 7t^2 - 5t + 4$ . In this case, we have

$$\begin{bmatrix} -5 & 17 & 12 \\ 17 & -60 & -32 \\ 12 & -32 & -64 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} =$$

$$\begin{bmatrix} -5 & 17 & 12 \\ 0 & 11 & -44 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

Again, the bottom right element is the value of the determinant, which verifies that the resultant is zero. It is now simple to compute the solution:  $X_1 = 4X_0$ ,  $X_2 = 4X_1$ . Since  $X_i = t^i$ , the common root is  $t = 4$ .

### Example 3

For our final example we analyze the polynomials  $f(t) = t^3 - 6t^2 + 11t - 6$  and  $g(t) = t^3 - 7t^2 + 14t - 8$ . Our linear equations now are:

$$\begin{bmatrix} -1 & 3 & -2 \\ 3 & -9 & 6 \\ -2 & 6 & -4 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} X_2 \\ X_1 \\ X_0 \end{Bmatrix} = 0$$

In this case, not only is the resultant zero, but the matrix is rank 1. This means that there are *two* common roots, and they can be found as the solution to the quadratic equation  $-t^2 + 3t - 2$ , which is  $t = 1$  and  $t = 2$ . Another way of saying this is that  $-t^2 + 3t - 2$  is the *Greatest Common Divisor* of  $f(t)$  and  $g(t)$ .

**Euclid's GCD Algorithm** An alternative approach to finding the common root(s) of two polynomials is to use Euclid's algorithm. This ancient algorithm can be used to find the *Greatest Common Divisor* of two integers or two polynomials. A clear proof of Euclid's algorithm can be found in [Kurosh '75]. Our presentation consists of a series of examples. This algorithm works beautifully in exact integer arithmetic but we have experienced numerical instability in floating point.

### Integer Example

We illustrate first on a pair of integers: 42 and 30. For the first step, we assign the larger to be the numerator, and the other to be the denominator:

Step 1:  $\frac{42}{30} = 1$  remainder 12.

We now take the remainder of the first step and divide it into the denominator of the first step:

Step 2:  $\frac{30}{12} = 2$  remainder 6.

We continue dividing the remainder of the preceding step into the denominator of the preceding step until we obtain a zero remainder. This happens to occur in the third step for this problem:

Step 3:  $\frac{12}{6} = 2$  exactly.

According to Euclid's algorithm, the second to last remainder is the GCD. In this case, the second to last remainder is 6, which is clearly the largest integer that evenly divides 30 and 42.

### Polynomial Example 1

We illustrate how Euclid's algorithm works for polynomials by using the same three examples we used in the previous section. For the polynomials  $f(t) = t^3 - 2t^2 + 3t + 1$  and  $g(t) = 2t^3 + 3t^2 - t + 4$ , we have:

- Step 1:  $\frac{2t^3 + 3t^2 - t + 4}{t^3 - 2t^2 + 3t + 1} = 2$  remainder  $7t^2 - 7t + 2$ .
- Step 2:  $\frac{t^3 - 2t^2 + 3t + 1}{7t^2 - 7t + 2} = \frac{t-1}{7}$  remainder  $\frac{12t+9}{7}$
- Step 3:  $\frac{7t^2 - 7t + 2}{(12t+9)/7} = \frac{196t - 343}{48}$  remainder  $\frac{1253}{16}$
- Step 4:  $\frac{(12t+9)/7}{1253/16} = \frac{192t}{8771} + \frac{144}{8771}$  remainder 0.

In this case, the GCD is  $\frac{1253}{16}$ , which is merely a constant, and so  $f(t)$  and  $g(t)$  do not have a common root.

### Polynomial Example 2

We next analyze the polynomials  $f(t) = t^3 - t^2 - 11t - 4$  and  $g(t) = 2t^3 - 7t^2 - 5t + 4$ :

- Step 1:  $\frac{2t^3 - 7t^2 - 5t + 4}{t^3 - t^2 - 11t - 4} = 2$  remainder  $-5t^2 + 17t + 12$ .
- Step 2:  $\frac{t^3 - t^2 - 11t - 4}{-5t^2 + 17t + 12} = -5t - \frac{12}{25}$  remainder  $\frac{-11t + 44}{25}$
- Step 3:  $\frac{-5t^2 + 17t + 12}{(-11t + 44)/25} = 125t + \frac{75}{11}$  remainder 0.

In this case, the GCD is  $\frac{-11t + 44}{25}$ , and the common root is  $t = 4$ .

### Polynomial Example 3

Finally, consider  $f(t) = t^3 - 6t^2 + 11t - 6$  and  $g(t) = t^3 - 7t^2 + 14t - 8$ :

- Step 1:  $\frac{t^3 - 6t^2 + 11t - 6}{t^3 - 7t^2 + 14t - 8} = 1$  remainder  $t^2 - 3t + 2$
- Step 2:  $\frac{t^3 - 7t^2 + 14t - 8}{t^2 - 3t + 2} = t - 4$  remainder 0.

The GCD is  $t^2 - 3t + 2$ , and the common roots are the roots of the equation  $t^2 - 3t + 2 = 0$  which are  $t = 1$  and  $t = 2$ .

You may have realized that there is a close connection between Euclid's algorithm and resultants, and obviously Euclid's algorithm does everything for us that resultants do.

We are now prepared to apply these tools to the problems of implicitizing and inverting curves.

## 17.5 Implicitization and Inversion

We discussed in the previous section a tool for determining whether two polynomials have a common root. We want to apply that tool to converting the parametric equation of a curve given by  $x = \frac{x(t)}{w(t)}$ ,  $y = \frac{y(t)}{w(t)}$  into an implicit equation of the form  $f(x, y) = 0$ . We proceed by forming two auxiliary polynomials:

$$p(x, t) = w(t)x - x(t) \quad q(y, t) = w(t)y - y(t)$$

Note that  $p(x, t) = q(y, t) = 0$  only for values of  $x$ ,  $y$ , and  $t$  which satisfy the relationships  $x = \frac{x(t)}{w(t)}$  and  $y = \frac{y(t)}{w(t)}$ . View  $p(x, t)$  as a polynomial in  $t$  whose coefficients are linear in  $x$ , and view  $q(y, t)$  as a polynomial in  $t$  whose coefficients are linear in  $y$ . If

$$x(t) = \sum_{i=0}^n a_i t^i, \quad y(t) = \sum_{i=0}^n b_i t^i, \quad w(t) = \sum_{i=0}^n d_i t^i$$

then

$$\begin{aligned} p(x, t) &= (d_n x - a_n) t^n + (d_{n-1} x - a_{n-1}) t^{n-1} + \dots \\ &\quad + (d_1 x - a_1) t + (d_0 x - a_0) \\ q(y, t) &= (d_n y - b_n) t^n + (d_{n-1} y - b_{n-1}) t^{n-1} + \dots \\ &\quad + (d_1 y - b_1) t + (d_0 y - b_0) \end{aligned}$$

If we now compute the resultant of  $p(x, t)$  and  $q(y, t)$ , we do not arrive at a numerical value, but rather a *polynomial* in  $x$  and  $y$  which we shall call  $f(x, y)$ . Clearly, any  $(x, y)$  pair for which  $f(x, y) = 0$  causes the resultant of  $p$  and  $q$  to be zero. But, if the resultant is zero, then we know that there exists a value of  $t$  for which  $p(x, t) = q(y, t) = 0$ . In other words, all  $(x, y)$  for which  $f(x, y) = 0$



lie on the parametric curve and therefore  $f(x, y) = 0$  is the implicit equation of that curve. This should be clarified by the following examples.

### Implicitization Example 1

Let's begin by applying this technique to the parabola we implicitized earlier using a brute force method:

$$x = t^2 + 1 \quad y = t^2 + 2t - 2.$$

We begin by forming  $p(x, t) = -t^2 + (x - 1)$  and  $q(y, t) = -t^2 - 2t + (y + 2)$ . The resultant of two quadratic polynomials  $a_2t^2 + a_1t + a_0$  and  $b_2t^2 + b_1t + b_0$  is

$$\begin{vmatrix} (a_2b_1) & (a_2b_0) \\ (a_1b_0) & (a_0b_0) \end{vmatrix}$$

and so the resultant of  $p(x, t)$  and  $q(y, t)$  is

$$R(p, q) = \begin{vmatrix} 2 & x - y - 3 \\ x - y - 3 & 2x - 2 \end{vmatrix} = -x^2 + 2xy - y^2 + 10x - 6y - 13$$

which is the implicit equation we arrived at earlier.

We can write an inversion equation for this curve – something which eluded us in our ad hoc approach:

$$\begin{bmatrix} 2 & x - y - 3 \\ x - y - 3 & 2x - 2 \end{bmatrix} \begin{Bmatrix} t \\ 1 \end{Bmatrix} = 0$$

From which  $t = \frac{-x + y + 3}{2}$  or  $t = \frac{-2x + 2}{x - y - 3}$ .

### Implicitization Example 2

We now implicitize the cubic curve for which

$$x = \frac{2t^3 - 18t^2 + 18t + 4}{-3t^2 + 3t + 1}$$

$$y = \frac{39t^3 - 69t^2 + 33t + 1}{-3t^2 + 3t + 1}$$

We begin by forming  $p(x, t)$  and  $q(y, t)$ :

$$p(x, t) = -2t^3 + (-3x + 18)t^2 + (3x - 18)t + (x - 4)$$

$$q(y, t) = -39t^3 + (-3y + 69)t^2 + (3y - 33)t + (y - 1)$$

Recalling from Section 17.3 that the resultant of two cubic polynomials  $a_3t^3 + a_2t^2 + a_1t + a_0$  and  $a_3t^3 + a_2t^2 + a_1t + a_0$  is

$$\begin{vmatrix} (a_3b_2) & (a_3b_1) & (a_3b_0) \\ (a_2b_1) & (a_2b_0) + (a_1b_1) & (a_2b_0) \\ (a_1b_0) & (a_0b_0) & (a_0b_0) \end{vmatrix},$$

we have

$$R(p, q) = f(x, y) = \begin{vmatrix} -117x + 69y + 564 & 117x - 6y - 636 & 39x - 2y - 154 \\ 117x - 6y - 636 & -69x - 2y + 494 & -66x + 6y + 258 \\ 39x - 2y - 154 & -66x - 2y + 258 & 30x - 6y - 114 \end{vmatrix}.$$

We can expand the determinant to get

$$\begin{aligned} f(x, y) = & -156195x^3 + 60426x^2y - 7056xy^2 + 224y^3 + \\ & 2188998x^2 - 562500xy + 33168y^2 - 10175796x + 1322088y + \\ & 15631624 \end{aligned}$$

We can obtain an inversion equation using Cramer's rule:

$$t = \frac{t^2}{t} = - \frac{\begin{vmatrix} (117x - 6y - 636) & (39x - 2y - 154) \\ (-69x - 2y + 494) & (-66x + 6y + 258) \end{vmatrix}}{\begin{vmatrix} (-117x + 69y + 564) & (39x - 2y - 154) \\ (117x - 6y - 636) & (-66x + 6y + 258) \end{vmatrix}}$$

Alternately, we could use Gauss elimination to compute the parameter of a point on the curve. Cramer's rule has the appeal that it actually generates an *equation*.

We have intentionally carried out all computations in exact integer arithmetic to emphasize the rational, non-iterative nature of implicitization and inversion. Since the coefficients of the implicit equation are obtained from the coefficients of the parametric equations using only multiplication, addition and subtraction, it is possible to obtain an implicit equation which *precisely* defines the same point set as is defined by the parametric equations.

## 17.6 Curve Inversion Made Simple

We now present a more simple approach to curve inversion. For a rational curve of degree  $n$ , we can generally write the inversion equation in the form:

$$t = \frac{f(x, y)}{g(x, y)} \tag{17.1}$$

where  $f(x, y)$  and  $g(x, y)$  are polynomials of degree  $n - 2$  if  $n > 2$ , and degree one if  $n = 2$ . Since we are mainly interested in curves of degree two and three,  $f(x, y)$  and  $g(x, y)$  are just degree one in those cases.

Suppose we want to find the inversion equation for a rational cubic curve (in homogeneous form)

$$x(t) = x_0 + x_1t + x_2t^2 + x_3t^3$$

$$y(t) = y_0 + y_1t + y_2t^2 + y_3t^3$$

$$w(t) = w_0 + w_1t + w_2t^2 + w_3t^3$$

Denote

$$f(x, y, w) = a_2x + b_2y + c_2w; \quad g(x, y, w) = a_1x + b_1y + c_1w. \tag{17.2}$$

To find an inversion equation for a cubic curve, we need just determine the coefficients  $a_1, a_2, b_1, b_2, c_1, c_2$  such that

$$\frac{f(x(t), y(t), w(t))}{g(x(t), y(t), w(t))} \equiv t \quad (17.3)$$

or

$$t \cdot g(x(t), y(t), w(t)) - f(x(t), y(t), w(t)) \equiv 0 \quad (17.4)$$

or

$$t \cdot [a_1(x_0 + x_1t + x_2t^2 + x_3t^3) + b_1(y_0 + y_1t + y_2t^2 + y_3t^3) + c_1(w_0 + w_1t + w_2t^2 + w_3t^3)] \\ - [a_2(x_0 + x_1t + x_2t^2 + x_3t^3) + b_2(y_0 + y_1t + y_2t^2 + y_3t^3) + c_2(w_0 + w_1t + w_2t^2 + w_3t^3)] \equiv 0$$

or

$$\begin{aligned} & t^4 \quad [a_1x_3 + b_1y_3 + c_1w_3] \\ +t^3 & [a_1x_2 + b_1y_2 + c_1w_2 + a_2x_3 + b_2y_3 + c_2w_3] \\ +t^2 & [a_1x_1 + b_1y_1 + c_1w_1 + a_2x_2 + b_2y_2 + c_2w_2] \\ +t^1 & [a_1x_0 + b_1y_0 + c_1w_0 + a_2x_1 + b_2y_1 + c_2w_1] \\ + & [a_2x_0 + b_2y_0 + c_2w_0] \equiv 0. \end{aligned} \quad (17.5)$$

This equation will be identically equal to zero if and only if all of the coefficients of the power of  $t$  are all zero. Thus, we can solve for the  $a_i, b_i$ , and  $c_i$  from the set of linear equations:

$$\begin{bmatrix} x_3 & y_3 & w_3 & 0 & 0 & 0 \\ x_2 & y_2 & w_2 & x_3 & y_3 & w_3 \\ x_1 & y_1 & w_1 & x_2 & y_2 & w_2 \\ x_0 & y_0 & w_0 & x_1 & y_1 & w_1 \\ 0 & 0 & 0 & x_0 & y_0 & w_0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (17.6)$$

Of course, degree two curves are even more simple. Note that we can also find inversion equations for 3D curves using this method.

## 17.7 Curve-Curve Intersections

Given one curve defined by the implicit equation  $f(x, y) = 0$  and a second curve defined by the parametric equations  $x = x(t)$ ,  $y = y(t)$ , we replace all occurrences of  $x$  in the implicit equation by  $x(t)$ , and replace all occurrences of  $y$  in the implicit equation by  $y(t)$ . These substitutions create a polynomial  $f(x(t), y(t)) = g(t)$  whose roots are the parameter values of the points of intersection. Of course, if we start off with two parametric curves, we can first implicitize one of them.

We illustrate this process by intersecting the curve

$$x = \frac{2t_1^3 - 18t_1^2 + 18t_1 + 4}{-3t_1^2 + 3t_1 + 1}$$

$$y = \frac{39t_1^3 - 69t_1^2 + 33t_1 + 1}{-3t_1^2 + 3t_1 + 1}$$

with the curve

$$x = \frac{-52t_2^3 + 63t_2^2 - 15t_2 + 7}{-37t_2^2 + 3t_2 + 1}$$

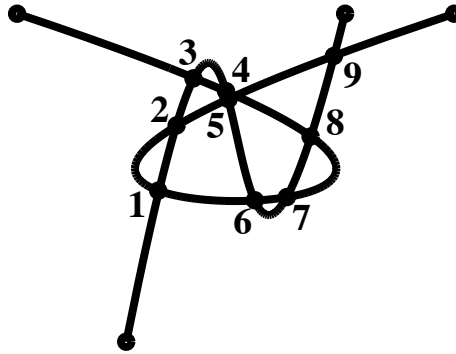


Figure 17.1: Two cubic curves intersecting nine times

$$y = \frac{4}{-37t_2^2 + 3t_2 + 1}$$

The two curves intersect nine times, which is the most that two cubic curves can intersect.<sup>2</sup> We already implicitized the first curve (in Section 17.3), so our intersection problem requires us to make the substitutions  $x = \frac{-52t_2^3 + 63t_2^2 - 15t_2 + 7}{-37t_2^2 + 3t_2 + 1}$  and  $y = \frac{4}{-37t_2^2 + 3t_2 + 1}$  into the implicit equation of curve 1:

$$\begin{aligned} f(x, y) = & -156195x^3 + 60426x^2y - 7056xy^2 + 224y^3 + \\ & 2188998x^2 - 562500xy + 33168y^2 - 10175796x + 1322088y + \\ & 15631624. \end{aligned}$$

After multiplying through by  $(-37t_2^2 + 3t_2 + 1)^3$ , we arrive at the intersection equation:

$$\begin{aligned} & 984100t_2^9 - 458200t_2^8 + 8868537t_2^7 - 9420593t_2^6 + 5949408t_2^5 \\ & - 2282850t_2^4 + 522890t_2^3 - 67572t_2^2 + 4401t_2 - 109 = 0. \end{aligned}$$

Again, we have carried out this process in exact integer arithmetic to emphasize that this equation is an *exact* representation of the intersection points.

We now compute the roots of this degree 9 polynomial. Those roots are the parameter values of the points of intersection. The  $(x, y)$  coordinates of those intersection points can be easily found from the parametric equation of the second curve, and the parameter values on the first curve for the intersection points can be found from the inversion equations. The results are tabulated below.

<sup>2</sup>Bezout's theorem states that two curves of degree  $m$  and  $n$  respectively, intersect in  $mn$  points, if we include complex points, points at infinity, and multiple intersections.

| Intersection Number | $t_1$<br>Parameter of Curve 1 | Coordinates of Point | $t_2$<br>Parameter of Curve 2 |
|---------------------|-------------------------------|----------------------|-------------------------------|
| 1                   | 0.0621                        | (4.2982,2.3787)      | 0.3489                        |
| 2                   | 0.1098                        | (4.4556,2.9718)      | 0.1330                        |
| 3                   | 0.1785                        | (4.6190,3.4127)      | 0.9389                        |
| 4                   | 0.3397                        | (4.9113,3.2894)      | 0.9219                        |
| 5                   | 0.4212                        | (4.9312,3.2186)      | 0.0889                        |
| 6                   | 0.6838                        | (5.1737,2.2902)      | 0.5339                        |
| 7                   | 0.8610                        | (5.4676,2.3212)      | 0.5944                        |
| 8                   | 0.9342                        | (5.6883,2.8773)      | 0.8463                        |
| 9                   | 0.9823                        | (5.9010,3.6148)      | 0.0369                        |

The most common curve intersection algorithms are currently based on subdivision. Tests indicate that this implicitization algorithm is faster than subdivision methods for curves of degree two and three, and subdivision methods are faster for curves of degree five and greater [Sederberg et al '86].

## 17.8 Surfaces

Implicitization and inversion algorithms exist for surfaces, also (see [Sederberg '83] or [Sederberg et al. '84b]). But, whereas curve implicitization yields implicit equations of the same degree as the parametric equations, surface implicitization experiences a degree explosion. A triangular surface patch, whose parametric equations are of the form

$$x = \frac{\sum_{i+j \leq n} x_{ij} s^i t^j}{\sum_{i+j \leq n} w_{ij} s^i t^j} \quad y = \frac{\sum_{i+j \leq n} y_{ij} s^i t^j}{\sum_{i+j \leq n} w_{ij} s^i t^j}$$

$$z = \frac{\sum_{i+j \leq n} z_{ij} s^i t^j}{\sum_{i+j \leq n} w_{ij} s^i t^j} \quad i, j \geq 0,$$

generally has an implicit equation of degree  $n^2$ . A tensor product surface patch, whose parametric equations are of the form

$$x = \frac{\sum_{i=0}^n \sum_{j=0}^m x_{ij} s^i t^j}{\sum_{i=0}^n \sum_{j=0}^m w_{ij} s^i t^j} \quad y = \frac{\sum_{i=0}^n \sum_{j=0}^m y_{ij} s^i t^j}{\sum_{i=0}^n \sum_{j=0}^m w_{ij} s^i t^j}$$

$$z = \frac{\sum_{i=0}^n \sum_{j=0}^m z_{ij} s^i t^j}{\sum_{i=0}^n \sum_{j=0}^m w_{ij} s^i t^j},$$

generally has an implicit equation of degree  $2mn$ . Thus, a bicubic patch generally has an implicit equation  $f(x, y, z) = 0$  of degree 18. Such an equation has 1330 terms!

Algebraic geometry shares important information on the nature of intersections of parametric surfaces. Recall that Bezout's theorem states that two surfaces of degree  $m$  and  $n$  respectively intersect in a curve of degree  $mn$ . Thus, two bicubic patches generally intersect in a curve of degree 324.

We have noted that bilinear patches have an implicit equation of degree 2; quadratic patches have an implicit equation of degree 4; biquadratic patches have an implicit equation of degree 8, etc. It seems highly curious that there are gaps in this sequence of degrees. Are there no parametric

surfaces whose implicit equation is degree 3 or 5 for example? It turns out that parametric surfaces of degree  $n$  only *generally* have implicit equations of degree  $n^2$  and that under certain conditions that degree will decrease. To understand the nature of those conditions, we must understand why the implicit equation of a parametric surface is generally  $n^2$ . The *degree* of a surface can be thought of either as the degree of its implicit equation, or as the number of times it is intersected by a line. Thus, the degree of the implicit equation of a parametric surface can be found by determining the number of times it is intersected by a line. Consider a parametric surface given by

$$x = \frac{x(s,t)}{w(s,t)} \quad y = \frac{y(s,t)}{w(s,t)} \quad z = \frac{z(s,t)}{w(s,t)},$$

where the polynomials are of degree  $n$ . One way we can compute the points at which a line intersects the surface is by intersecting the surface with two planes which contain the line. If one plane is  $Ax + By + Cz + Dw = 0$ , its intersection with the surface is a curve of degree  $n$  in  $s, t$  space:  $Ax(s,t) + By(s,t) + Cz(s,t) + Dw(s,t) = 0$ . The second plane will also intersect the surface in a degree  $n$  curve in parameter space. The points at which these two section curves intersect will be the points at which the line intersects the surface. According to Bezout's theorem, two curves of degree  $n$  intersect in  $n^2$  points. Thus, the surface is generally of degree  $n^2$ .

## 17.9 Base Points

It may happen that there are values of  $s, t$  for which  $x(s,t) = y(s,t) = z(s,t) = w(s,t) = 0$ . These are known as *base points* in contemporary algebraic geometry. If a base point exists, any plane section curve will contain it. Therefore, it will belong to the set of intersection points of any pair of section curves. However, since a base point maps to something that is undefined in  $x, y, z$  space, it does not represent a point at which the straight line intersects the surface, and thus *the existence of a base point diminishes the degree of the implicit equation by one*. Thus, if there happen to be  $r$  base points on a degree  $n$  parametric surface, the degree of its implicit equation is  $n^2 - r$ .

For example, it is well known that any quadric surface can be expressed in terms of degree 2 parametric equations. However, in general, a degree 2 parametric surface has an implicit equation of degree 4, and we conclude that there must be two base points. Consider the parametric equations of a sphere of radius  $r$  centered at the origin:

$$\begin{aligned} x &= 2rs^u \\ y &= 2rtu \\ z &= r(u^2 - s^2 - t^2) \\ w &= s^2 + t^2 + u^2 \end{aligned}$$

Homogeneous parameters  $s, t, u$  are used to enable us to verify the existence of the two base points:  $s = 1, t = i, u = 0$  and  $s = 1, t = -i, u = 0$ .

## 17.10 Implicitization in Bézier Form

From the paper [SP86a], a Bézier curve can be implicitized as follows. (Note that the value  $l_{ij}$  in these notes is equivalent to the value  $L_{j,k+1}$  in the paper).

A degree 2 Bézier curve can be implicitized:

$$f(x, y) = \begin{vmatrix} l_{01}(x, y) & l_{02}(x, y) \\ l_{02}(x, y) & l_{12}(x, y) \end{vmatrix}$$

and a degree 3 Bézier curve can be implicitized

$$f(x, y) = \begin{vmatrix} l_{01}(x, y) & l_{02}(x, y) & l_{03}(x, y) \\ l_{02}(x, y) & l_{03}(x, y) + l_{12}(x, y) & l_{13}(x, y) \\ l_{03}(x, y) & l_{13}(x, y) & l_{23}(x, y) \end{vmatrix}$$

where

$$l_{ij}(x, y) = \binom{n}{i} \binom{n}{j} w_i w_j \begin{vmatrix} x & y & 1 \\ x_i & y_i & 1 \\ x_j & y_j & 1 \end{vmatrix}$$

with  $n$  the degree of the curve, and  $x_i, y_i, w_i$  the coordinates and weight of the  $i^{\text{th}}$  control point.

For a general degree  $n$  curve, the implicit equation is

$$f(x, y) = \begin{vmatrix} L_{0,0}(x, y) & \cdot & \cdot & \cdot & L_{0,n-1}(x, y) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ L_{n-1,0}(x, y) & \cdot & \cdot & \cdot & L_{n-1,n-1}(x, y) \end{vmatrix}$$

where

$$L_{i,j} = \sum_{\substack{k \leq \min(i,j) \\ k+m=i+j+1}} l_{km}.$$

Inversion is accomplished by solving the set of equations

$$\begin{bmatrix} l_{01}(x, y) & l_{02}(x, y) & l_{03}(x, y) \\ l_{02}(x, y) & l_{03}(x, y) + l_{12}(x, y) & l_{13}(x, y) \\ l_{03}(x, y) & l_{13}(x, y) & l_{23}(x, y) \end{bmatrix} \begin{Bmatrix} (1-t)^2 \\ t(1-t) \\ t^2 \end{Bmatrix} = 0$$

for the degree three case, and

$$\begin{bmatrix} l_{01}(x, y) & l_{02}(x, y) \\ l_{02}(x, y) & l_{12}(x, y) \end{bmatrix} \begin{Bmatrix} (1-t) \\ t \end{Bmatrix} = 0$$

for degree two.

## 17.11 Gröbner Bases

Gröbner bases provide a powerful approach to solving systems of polynomial equations.

### 17.11.1 Notation and terminology

In general, a polynomial in  $n$  variables  $x_1, \dots, x_n$  is defined

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{\tau} c_i x_1^{e_{1,i}} x_2^{e_{2,i}} \dots x_n^{e_{n,i}}. \quad (17.7)$$

Each summand  $c_i x_1^{e_{1,i}} x_2^{e_{2,i}} \cdots x_n^{e_{n,i}}$  is called a *term*,  $x_1^{e_{1,i}} x_2^{e_{2,i}} \cdots x_n^{e_{n,i}}$  is a *monomial*, and  $c_i$  is the *coefficient* of the monomial. By convention, any given monomial occurs in at most one term in a polynomial.

$k[x_1, \dots, x_n]$  signifies the set of all polynomials in the variables  $x_1, \dots, x_n$  whose coefficients belong to a field  $k$ . For example,  $R[x, y]$  is the set of all polynomials

$$\sum c_i x^{e_{1,i}} y^{e_{2,i}} \quad (17.8)$$

where  $c_i \in R$  and  $e_{1,i}, e_{2,i} \in \{0, 1, 2, \dots\}$ . Thus, “ $f \in R[x, y, z]$ ” means that  $f$  is a polynomial whose variables are  $x, y$  and  $z$  and whose coefficients are real numbers. All polynomials in this chapter have coefficients that are real numbers.

### Term order

It is often useful to list the terms of a polynomial in decreasing order, beginning with the *leading term*. This is done using a *term order* — a way to compare any two distinct terms of a polynomial and declare which is “greater.”

For linear polynomials, term order amounts to merely declaring an order on the variables. For example, the terms of the polynomial

$$2x + 3y - 4z$$

are in proper order if we declare  $x > y > z$ . If we declare  $y > z > x$ , the proper order would be  $3y - 4z + 2x$ . For non-linear polynomials, we begin by declaring an order on the variables, and then we must also choose one of several schemes that decide how the exponents in a polynomial influence term order. One such scheme is called *lexicographical order* (nicknamed *lex*), defined as follows. If the variables of a polynomial are ordered  $x_1 > x_2 > \dots > x_n$ , then given two distinct terms  $T_i = c_i x_1^{e_{1,i}} x_2^{e_{2,i}} \cdots x_n^{e_{n,i}}$  and  $T_j = c_j x_1^{e_{1,j}} x_2^{e_{2,j}} \cdots x_n^{e_{n,j}}$ ,  $T_i > T_j$  if

1.  $e_{1,i} > e_{1,j}$ , or if
2.  $e_{1,i} = e_{1,j}$  and  $e_{2,i} > e_{2,j}$ , or, in general, if
3.  $e_{k,i} = e_{k,j}$  for  $k = 1, \dots, m-1$  and  $e_{m,i} > e_{m,j}$ .

For example, the polynomial

$$3x^2y^2z + 4xy^3z^2 + 5x^3z + 6y^2 + 7xz^3 + 8$$

using *lex* with  $x > y > z$  would be written  $5x^3z + 3x^2y^2z + 4xy^3z^2 + 7xz^3 + 6y^2 + 8$  and its leading term is  $5x^3z$ . Using *lex* with  $z > x > y$  it would be written  $7z^3x + 4z^2xy^3 + 5zx^3 + 3zx^2y^2 + 6y^2 + 8$  and the leading term would be  $7z^3x$ . Or using *lex* with  $y > z > x$  would be written  $4y^3z^2x + 3y^2zx^2 + 6y^2 + 7z^3x + 5zx^3 + 8$  and the leading term would be  $4y^3z^2x$ .

Another choice for term order is the *degree lexicographical order* (abbreviated *deglex*). If the variables are ordered  $x_1 > x_2 > \dots > x_n$ , then using *deglex*,  $T_i > T_j$  if

1.  $e_{1,i} + e_{2,i} + \dots + e_{n,i} > e_{1,j} + e_{2,j} + \dots + e_{n,j}$ , or
2.  $e_{1,i} + e_{2,i} + \dots + e_{n,i} = e_{1,j} + e_{2,j} + \dots + e_{n,j}$  and  $T_i > T_j$  with respect to *lex*.

Using *deglex* with  $x > y > z$ , the terms of  $3x^2y^2z + 4xy^3z^2 + 5x^3z + 6y^2 + 7xz^3 + 8$  would be ordered  $4xy^3z^2 + 3x^2y^2z + 5x^3z + 7xz^3 + 6y^2 + 8$ .



As observed in the lex and deglex examples, term orders ignore the coefficient of a term, so a term order might more properly be called a monomial order.

Other term orders can also be defined, such as degree reverse lexicographical order. The precise requirements for any term order are discussed in reference [AL94], page 18.

The  $n$ -dimensional real affine space is denoted  $R^n$  and is the set of  $n$ -tuples:

$$R^n = (a_1, \dots, a_n) : a_1, \dots, a_k \in R \quad (17.9)$$

## 17.12 Ideals and Varieties

This section presents a brief overview of ideals and varieties and suggests some ways how these topics fit into CAGD. An excellent treatment of ideals and varieties and their application to CAGD can be found in [CLO92].

The *polynomial ideal* generated by  $f_1, \dots, f_s \in k[x_1, \dots, x_n]$ , denoted  $\langle f_1, \dots, f_s \rangle$ , is defined

$$\langle f_1, \dots, f_s \rangle = \{p_1 f_1 + \dots + p_s f_s : p_i \in k[x_1, \dots, x_n]\}.$$

Consider a set of polynomials  $f_1, f_2, \dots, f_s \in k[x_1, \dots, x_n]$ . Let  $(a_1, \dots, a_n)$  be a point in  $k^n$  satisfying  $f_i(a_1, \dots, a_n) = 0$ ,  $i = 1, \dots, s$ . The set of all such points  $(a_1, \dots, a_n)$  is called the *variety* defined by  $f_1, \dots, f_s$ , and is denoted by  $V(f_1, \dots, f_s)$ :

$$V(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in k^n \mid f_i(a_1, \dots, a_n) = 0, i = 1, \dots, s\}. \quad (17.10)$$

A variety defined by a single polynomial—called a *hypersurface*—is the most familiar instance of a variety. A hypersurface in  $R^2$  is a planar curve defined using an implicit equation, and a hypersurface in  $R^3$  is what is normally called an implicit surface in CAGD. For example,  $V(x^2 + y^2 - 1)$  is a circle defined in terms of the implicit equation  $x^2 + y^2 - 1 = 0$  and  $V(2x + 4y - z + 1)$  is the plane whose implicit equation is  $2x + 4y - z + 1 = 0$ .

A variety  $V(f_1, \dots, f_s)$  defined by more than one polynomial ( $s > 1$ ) is the intersection of the varieties  $V(f_1) \dots V(f_s)$ .

### 17.12.1 Gröbner Bases

It can be very useful to devise alternative generators for an ideal. Necessary and sufficient conditions for  $\langle f_1, \dots, f_n \rangle = \langle g_1, \dots, g_m \rangle$  are  $f_1, \dots, f_n \in \langle g_1, \dots, g_m \rangle$  and  $g_1, \dots, g_m \in \langle f_1, \dots, f_n \rangle$ .

A *Gröbner basis* of an ideal  $I$  is a set of polynomials  $\{g_1, \dots, g_t\}$  such that the leading term of any polynomial in  $I$  is divisible by the leading term of at least one of the polynomials  $g_1, \dots, g_t$ . This, of course, requires that a term order be fixed for determining the leading terms: different term orders produce different Gröbner bases. Several excellent books have been written on Gröbner bases that do not presuppose that the reader has an advanced degree in mathematics [CLO92, AL94, BW93].

A Gröbner basis is a particularly attractive set of generators for an ideal, as illustrated by two familiar examples. If  $\{f_1, \dots, f_s\}$  are polynomials in one variable, the Gröbner basis of  $\langle f_1, \dots, f_n \rangle$  consists of a single polynomial: the GCD of  $f_1, \dots, f_s$ . If  $\{f_1, \dots, f_s\}$  are linear polynomials in several variables, the Gröbner basis is an uppertriangular form of a set of linear equations. The Gröbner basis of these special cases provides significant computational advantage and greater insight, and the same is true of the Gröbner basis of a more general ideal.

Gröbner bases are the fruit of Bruno Buchberger's Ph.D. thesis [Buc65], and are named in honor of his thesis advisor. Buchberger devised an algorithm for computing Gröbner bases [Buc85, CLO92]. Also, commercial software packages such as Maple and Mathematica include capabilities for computing Gröbner bases.

