THE DISTANCE FOR THE BÉZIER CURVES AND DEGREE REDUCTION

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Abstract. An algorithmic approach to degree reduction of Bézier curves is presented. The algorithm is based on the matrix representations of the degree elevation and degree reduction processes. The control points of the approximation are obtained by the generalized least square method. The computations are carried out by minimizing the L_2 and discrete l_2 distance between the two curves.

1. Introduction

The Bézier curves are basically and widely used in CAGD – short for Computer Aided Geometric Design. The Bézier curves were independently developed by P. de Casteljau about 1959[2] and by P. Bézier about 1962[1]. The underlying mathematical theory is based on the concept of Bernstein polynomials. De Casteljau directly exploited this relationship; but it was not before 1970 that R. Forrest[11] discovered the connection between Bézier's work and Bernstein polynomials. Bézier and de Casteljau developed their theories as part of CAD systems that were being built up at two French car companies, Rénault and Citroën. The Rénault system UNISURF (by Bézier) was soon described in several publications; this is the reason that the underlying theory now bears Bézier's name. Bézier curves and surfaces are now established as the mathematical basis of many CAD systems, they have also become a major tool for the development of new methods for curve and surface descriptions. Farin[10] summarize the basic theory of such curves and provide many relevant references.

The Bézier representation uses Bernstein polynomials as basis functions for the linear space of polynomials. In terms of the Bernstein polynomials of degree n,

$$B_k^n(t) = \binom{n}{k}(1-t)^{n-k}t^k,\ 0 \le t \le 1,\ k = 0, \cdots, n,$$

a parametric polynomial curve of degree n ($n > 0$) in the plane, can be expressed as

$$b^n(t) = \sum_{k=0}^{n} b_k B_k^n(t),\ b_k \in \mathbb{R}^2.$$

The points b_k, $k = 0, \cdots, n$ are called the *control points* for the polynomial, and the polygon formed by joining successive control points is the *control polygon*. Notice that b_0 and b_n are the endpoints of the curve corresponding to $t = 0$ and $t = 1$; we shall refer to these particular points as *anchor points*. Moreover, the vector $b_1 - b_0$ and $b_n - b_{n-1}$ define the tangents to the curve at the two anchor points respectively.

In general degree reduction of Bézier curves address the following problem.

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Problem 1 (Degree Reduction). *Let* $\{b_i\}_{i=0}^{n} \subset \mathbb{R}^2$ *be a given set of control points which define the Bézier curve*

$$b^n(t) = \sum_{i=0}^{n} b_i B_i^n(t), \quad 0 \le t \le 1$$

of degree n*. Then find another point set* $\{c_i\}_{i=0}^{m} \subset \mathbb{R}^2$ *defining the approximative Bézier curve*

$$c^m(t) = \sum_{i=0}^{m} c_i B_i^m(t), \quad 0 \le t \le 1$$

of lower degree $m < n$ *so that a suitable distance function* $d(b^n, c^m)$ *between* b^n *and* c^m *is minimized.*

In the literatures([3], [4], [6], [7], [9], [11], [14], [15], [17], [19]) one can find several schemes producing solutions for this approximation problem. These schemes mainly differ in the choice of the distance function and requiring the solution to be either best or only nearly best relative to the distance function. For instance, one special type of degree reduction schemes works recursively by lowering the degree only by one in every step – a procedure commonly known as economization.

The examples for such a stepwise method was recently given in Eck[6] or Park, Choi and Kimn[19] where a very simple geometric construction of the new control points in each step is described. And the method allow the detailed error analysis for the other method (e.g. Forrest[11] and Farin[9], see Park and Choi[18]). However, this general construction contains some scalar – valued degrees of freedom which are then chosen in such a way that the maximal Euclidean distance

$$d_\infty(b^n, c^{n-1}) = \max_{0 \le t \le 1} \|b^n(t) - c^{n-1}(t)\|$$

between two curves with respect to the given parameterization is minizied.

The derivation is mainly based on the so–called constrained Chebyshev polynomials. Unfortunately, the constrained Chebyshev polynomials are not known explicitly so their coefficients have to be determined numerically, which itself needs a lot of implementation effort.

This major disadvantage is avoided in the current paper. In more detail, we minizie the least squares distance function

$$d_2(b^n, c^m) = \sqrt{\int_0^1 \|b^n(t) - c^m(t)\|^2 dt}.$$

The algorithm presented is faster, more stable and much easier to implement. Moreover, the procedure apply to reduce the degree from n to m as only one step.

2. Degree elevation and L_2 distance

Suppose we were designing with Bézier curves trying to use a Bézier curve of degree n. After modifying the polygon a few times, it may turn out that a degree n curve does not possess sufficient flexibility to model the desired shape. One way to proceed in such situation is to increase the flexibility of the polygon by adding another vertex to it. As a first step, one might want to another vertex yet leave the shape of the curve unchanged – this corresponds to raising the degree of the Bézier curve by one. We can show that new vertices $b_i^{(1)}$ are obtained from the old polygon by piecewise linear interpolation at the parameter values $i/(n+1)$

$$b_i^{(1)} = \frac{i}{n+1} b_{i-1} + (1 - \frac{i}{n+1}) b_i, i = 0, 1, \ldots, n+1. \tag{1}$$

We can rewrite the formular (1) as a linear system $T_n B = B^{(1)}$, where the $(n+2)\times(n+1)$ matrix T_n is

$$T_n = \frac{1}{n+1} \begin{pmatrix} n+1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & n & 0 & \ldots & 0 & 0 & 0 \\ 0 & 2 & n-1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & n-1 & 2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & n & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & n+1 \end{pmatrix}$$

and the $(n+1)$ vector B and the $(n+2)$ vector $B^{(1)}$ are

$$\begin{aligned} B &= (b_0, b_1, \ldots, b_n)^t \\ B^{(1)} &= (b_0^{(1)}, b_1^{(1)}, \ldots, b_n^{(1)})^t. \end{aligned}$$

We may repeat this process and then obtain a sequence of controls points. After r degree elevations, we have a linear system $T_{n,r} B = B^{(r)}$, where the $(n+r+1) \times (n+1)$ matrix

$$T_{n,r} = T_{n+r-1} T_{n+r-2} \ldots T_{n+1} T_n$$

has elements

$$t_{i+j,i} = \frac{\binom{n}{i}\binom{r}{j}}{\binom{n+r}{i+j}}, \quad \begin{cases} i = 0, 1, \ldots, n \\ j = 0, 1, \ldots, r \end{cases}$$

The sum of any row and any column of the matrix $T_{n,r}$ are 1 and $\frac{n+r+1}{n+1}$ respectively, i.e. for any i,

$$\sum_{k=0}^{n+r} t_{i,k} = 1,$$

and for any k,

$$\sum_{i=0}^{n} t_{i,k} = \frac{n+r+1}{n+1}.$$

For the degree reduction of any given curves, we must compute a distance of two Bézier curves. The most appropriate metric in geometrical terms would be the *Hausdorff distance*[5]. Suppose (M, d) is a metric space with subsets A and B. We define the Hausdorff metric d_H by

$$d_H(A, B) = \max\left\{ \sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A) \right\},$$

where

$$d(x, B) = \inf_{y\in B} d(x, y).$$

If we regard a planar curve as simply a locus of points without any underlying parameterization, the Hausdorff metric for two such curves is essentially the radius of the largest circle with its center on one curve and touching the other curve. For general parametric curves, this measure is truly independent of the relative parameterizations of two curves. Emery[8] presents a method for explicit computation of Hausdorff metric for piecewise linear curves, but the computation of Hausdorff distance d_H of two nonlinear curves is not so easy. So we define and use the L_2 distance for the Bézier curves.

We first consider the functional case of the Bézier curves for computation of L_2 distance of the two Bézier curves. Let a^n and b^m be the functional Bézier curves of degree n and m $(m < n)$, i.e.

$$a^n(t) = \sum_{k=0}^{n} a_k B_k^n(t) \ , \ b^m(t) = \sum_{k=0}^{m} b_k B_k^m(t) \ ,$$

where the coefficients a_k and b_k are real numbers. The L_2 distance of the two Bézier curves a^n and b^m is defined as following:

$$\begin{aligned} d_2(a^n, b^m) &= \left\{ \int_0^1 |a^n(t) - b^m(t)|^2 \, dt \right\}^{\frac{1}{2}} \\ &= \left\{ \int_0^1 \left| \sum_{k=0}^{n} a_k B_k^n(t) - \sum_{k=0}^{m} b_k B_k^m(t) \right|^2 dt \right\}^{\frac{1}{2}} \end{aligned}$$

Using the matrix $T_{m,r}$, we can elevate the degree of b^m from m to n,

$$B^{(r)} = T_{m,r} B.$$

Then, the curve b^m of degree m is re–written as a curve of degree n

$$b^m(t) = b^{(r)}(t) = \sum_{k=0}^{n} b_k^{(r)} B_k^n(t),$$

and the distance is

$$\begin{aligned} d_2(a^n, b^m) &= d_2(a^n, b^{(r)}) \\ &= \left\{ \int_0^1 \left| \sum_{k=0}^{n} a_k B_k^n(t) - \sum_{k=0}^{n} b_k^{(r)} B_k^n(t) \right|^2 dt \right\}^{\frac{1}{2}} \\ &= \left\{ \int_0^1 \left| \sum_{k=0}^{n} (a_k - b_k^{(r)}) B_k^n(t) \right|^2 dt \right\}^{\frac{1}{2}}. \end{aligned}$$

Let $c_k = a_k - b_k^{(r)}$ for all k. So, we compute the L_2 norm of a (functional) Bézier curve of degree n.

The product of Bernstein polynomials is

$$B_i^n(t) B_j^m(t) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}} B_{i+j}^{n+m}(t) \tag{2}$$

and the integration is

$$\int_0^1 B_k^n(t) dt = \frac{1}{n+1}. \tag{3}$$

¿From these equations (2) and (3), we obtain the following computation for the L_2 norm of the functional Bézier curve c^n:

$$\begin{aligned}
\|c^n\|_2{}^2 &= \int_0^1 \left| \sum_{k=0}^{n} c_k B_k^n(t) \right|^2 dt \\
&= \int_0^1 \sum_{i,j} c_i c_j B_i^n(t) B_j^n(t) dt \\
&= \sum_{i,j} c_i c_j \int_0^1 \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}} B_{i+j}^{2n}(t) dt \\
&= \frac{1}{2n+1} \sum_{i,j} c_i c_j \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}}
\end{aligned}$$

Let $(n+1) \times (n+1)$ matrix Q_n be the

$$Q_n = \frac{1}{2n+1} \left[\frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{i+j}} \right]_{(n+1)\times(n+1)}$$

Then, L_2 norm of the Bézier curve c^n is

$$\|c^n\|_2{}^2 = C^t Q_n C.$$

The matrix Q_n is a real symmetric matrix. The following lemma tell us the equivalent conditions for the real symmetric matrix to be positive definite matrix[16].

LEMMA 1. *Each of the following tests is a necessary and sufficient condition for the real symmetric matrix A to be positive definite:*

1. $\mathrm{x}^t A \mathrm{x} > 0$ *for all nonzero vectors* x.
2. *All the eigenvalues of A are positive.*
3. *All the upper left submatrices have positive determinants.*

¿From the definition of the matix Q_n and the mathematical induction, all the upper left submatrices of the matrix Q_n have positive determinants. Applying the Lemma 1, the matrix Q_n is a real symmetric positive definite.

Thus, we obtaine the following theorem for the L_2 distance between the Bézier curves a^n of degree n and the Bézier curve b^m of degree m.

THEOREM 2. *The L_2 distance between the two Bézier curves a^n and b^m is*

$$d_2(a^n, b^m) = d_2(a^n, b^{(r)}) = \sqrt{D^t Q_n D}, \tag{4}$$

where $D = A - T_{m,r} B$ and $A = (a_0, \ldots, a_n)^t$ and $B = (b_0, \ldots, b_m)^t$.

The sum of any row and any column of the matrix Q_n are both equal to $\frac{1}{n+1}$ i.e.

$$\sum_{j=0}^{n} q_{i,j} = \sum_{i=0}^{n} q_{i,j} = \frac{1}{n+1}.$$

3. Degree reduction

Replacing the distance function d in the Problem 1 and the distance d_2 as (4), then the Problem 1 rewritten as the following:

PROBLEM 2 (L_2 Degree Reduction). *Find another points set* $\{c_i\}_{i=0}^m$ *so that the least squares distance*

$$d_2(b^n, c^m) = d_2(b^n, c^{(r)}) = \sqrt{D^t Q_n D}$$

between $\{b_i\}_{i=0}^n$ *and* $\{c_i^{(r)}\}_{i=0}^n$*is minimized.*

Note that $T_{m,r}C = C^{(r)}$ and $B - C^{(r)} = D$ where $D = (d_0, d_1, \cdots, d_n)^t$, $d_i = b_i - c_i^{(r)}$, $i = 0, 1, \cdots, n$.

For developing the method rewrite the $D^t Q_n D$,

$$\begin{aligned} D^t Q_n D &= [B - C^{(r)}]^t Q_n [B - C^{(r)}] \\ &= [B - T_{m,r}C]^t Q_n [B - T_{m,r}C] \\ &= B^t Q_n B - 2C^t T_{m,r}^t Q_n B + C^t T_{m,r}^t Q_n T_{m,r} C. \end{aligned}$$

One method of obtaining the vector C is so-called method of least squares. This method consists of minimizing $D^t Q_n D$ with respect to C. Choosing the vector $\hat{C}$ that value of C which minimize $D^t Q_n D$ involves differentiating $D^t Q_n D$ with respect to the elements of C. Equating $\partial(D^t Q_n D)/\partial C$ to zero and writing the resulting equations in terms of $\hat{C}$, we find that these equations are

$$T_{m,r}^t Q_n T_{m,r} \hat{C} = T_{m,r}^t Q_n B.$$

They are known as the normal equations[12].

¿From the definition of the matrix $T_{m,r}$ and Q_n, we have that the matrix product $T_{m,r}^t Q_n T_{m,r}$ is Q_m i.e.

$$T_{m,r}^t Q_n T_{m,r} = Q_m.$$

Hence, the real symmetric positive definite matrix $T_{m,r}^t Q_n T_{m,r} = Q_m$ is ivertible. Provided $(T_{m,r}^t Q_n T_{m,r})^{-1}$ exists they have the unique solution for $\hat{C}$,

$$\hat{C} = (T_{m,r}^t Q_n T_{m,r})^{-1} T_{m,r}^t Q_n B. \tag{5}$$

The approximate curve by using (5) is best approximation with respect to L_2 norm. The L_2 best approximation is known as Legendre polynomials. See Eck[7] for detailed discussion.

For error analysis, we need the definition of the *Moore – Penrose inverse.*

DEFINITION 1. *Let A be the $n \times m (n > m)$ matrix. The $m \times n$ matrix X is called the Moore – Penrose inverse of the A, if it satisfies the following conditions known as* Moore – Penrose conditions*:*

$$AXA = A, \quad (AX)^t = AX$$
$$XAX = X, \quad (XA)^t = XA$$

The Moore – Penrose inverse of A is usually denoted by A^+. The Moore – Penrose inverse A^+ of the matrix A is uniquely determined[13]. If $rank(A) = m$, then $A^+ = (A^t A)^{-1} A^t$, while if $rank(A) = m = n$, then $A^+ = A^{-1}$[13].

The matrix $M = (T_{m,r}^t Q_n T_{m,r})^{-1} T_{m,r}^t Q_n$ is the Moore – Penrose inverse of the matrix $T_{m,r}$.

To obtain the approximation error ϵ_{L_2}, put $\hat{C}$ into the equation in the Problem 2.

THEOREM 3. *The error of the solution $\hat{C}$ of the Problem 2 is*

$$\epsilon_{L_2}^2 = B^t Q_n [I - T_{m,r}(T_{m,r}^t Q_n T_{m,r})^{-1} T_{m,r}^t] B$$

For simple computation we may use the discrete l_2 distance function, then the Problem 1 rewritten as the following:

PROBLEM 3 (l_2 Degree Reduction). *Find control points* $\{c_i\}_{i=0}^m$ *so that the distance*

$$d_{DLS}(b^n, c^m) = d_{DLS}(b^n, c^{(r)}) = \sqrt{D^t D}$$

between $\{b_i\}_{i=0}^n$ *and* $\{c_i^{(r)}\}_{i=0}^n$ *is minimized.*

As in the case of L_2 degree reduction, we obtain the solution $\hat{C}_{DLS}$ as

$$\hat{C}_{DLS} = (T_{m,r}^t T_{m,r})^{-1} T_{m,r}^t B$$

The matrix $(T_{m,r}^t T_{m,r})^{-1} T_{m,r}^t$ is also the Moore – Penrose inverse of $T_{m,r}$. By the uniqueness of the Moore – Penrose inverse, the L_2 solution $\hat{C}$ and the discrete l_2 solution $\hat{C}_{DLS}$ are equal, i.e.

$$\hat{C} = \hat{C}_{DLS}.$$

Thus the L_2 degree reduction curve and the discrete l_2 degree reduction curve are same.

To obtain the approximation error ϵ_{l_2}, put $\hat{C}_{DLS}$ into the equation in the Problem 3.

THEOREM 4. *The error of the solution* $\hat{C}_{DLS}$ *of the Problem 3 is*

$$\epsilon_{l_2}^2 = B^t [I - T_{m,r}(T_{m,r}^t T_{m,r})^{-1} T_{m,r}^t] B$$

The $(m+2)*(m+2)$ matrix $P_m = I - T_m(T_m^t T_m)^{-1} T_m^t = \{p_{i,j}\}$ has elements

$$p_{i,j} = (-1)^{i+j} \frac{\binom{m+1}{i}\binom{m+1}{j}}{\binom{2m+2}{m+1}}, i, j = 0, 1, \ldots, m+1.$$

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